The augmentation variety and the Gromov-Witten disk potential

Tobias Ekholm

Uppsala University and Institut Mittag-Leffler, Sweden

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Plan

- Large N transition
- Asymptotics of the wave function and the GW-disk potential
- Exact Lagrangian fillings and augmentations
- Non-exact Lagrangian fillings and augmentations
- Other manifolds

Recap

- Recall from yesterday, with $X = (T^*S^3, N \cdot S^3)$

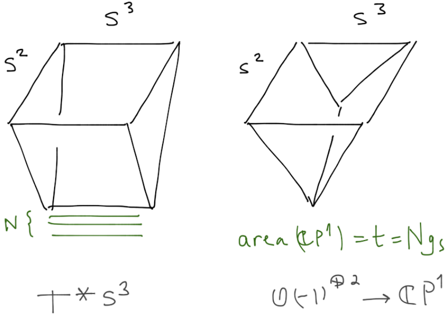
$$\begin{aligned}\Psi_K(x) &:= Z_{GW}(X, L_K)/Z_{GW}(X) \\ &= \sum_k \langle \mathrm{tr}_{S_k} U(K)\rangle e^{-kx} = \sum_k H_k(K)\, e^{-kx},\end{aligned}$$

where $H_k(K)$ is a HOMFLY-invariant which is polynomial in $q = e^{g_s}$ and $Q = q^N$,

$$H_k(K) = Z_{CS}(S^3; N, k)^{-1} \int \mathcal{D}A\, e^{\frac{ik}{4\pi} S(A)}\, \mathrm{tr}_{S_k}(\mathrm{Hol}_K(A)).$$

Large N transition

- $X = T^*S^3$ is a quadric in $\mathbb{C}^4$, a resolution of a cone. There is another resolution Y, total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{C}P^1$.

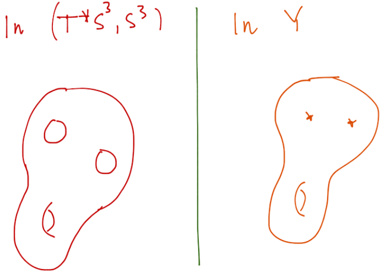


Large N transition

- Gopakumar-Vafa proposes: if area$(\mathbb{C}P^1) = t = Ng_s$, and $Q = e^t = q^N$, then

$$Z_{GW}(X; N, g_s) = Z_{GW}(Y; g_s, Q),$$

relating A-model open strings in X to A-model closed strings in Y.



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- L_K can be shifted off the 0-section by a non-exact Lagrangian isotopy. Thus, $L_K \subset Y$, and analogously

$$\Psi_K(x) = Z_{GW}(Y; L_K)/Z_{GW}(Y)$$

Physics results

- Witten's argument relates constant curves on L_K to $GL(1)$-gauge theory on the solid torus which corresponds to ordinary QM in the periods of the connection. This gives

$$p\Psi_K(x) = g_s \frac{\partial}{\partial x} \Psi_K(x),$$

and the usual asymptotics:

$$\Psi_K(x) = \exp\left(\frac{1}{g_s}\int p dx + \dots\right).$$

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- From the GW-perspective

$$\Psi_K(x) = \exp\left(\frac{1}{g_s} W_K(x) + \dots\right),$$

where $W_K(x)$ is the disk potential of L_K and where $\dots$ counts curves with $2g - 2 + h > -1$.

Physics results

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- The colored HOMFLY polynomial satisfy a recursion relation (is q-holonomic) and the above curve describes that relation on the level of the disk and should thus correspond to an algebraic curve V_K, given by a polynomial equation,

$$A_K(e^x, e^p, Q) = 0,$$

and in analogy with SYZ

$$A_K = uv$$

should be a mirror of Y with L_K corresponding to $p \times \{v = 0\}$, $p \in V_K$.

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- An augmentation of $\mathcal{A}$ is a chain map $\epsilon\colon \mathcal{A} \to \mathbb{C}$, $\epsilon \circ \partial = 1$, of DGAs, where we consider $\mathbb{C}$ as the trivial DGA in degree 0.
- The augmentation variety V_K is the algebraic closure of

$$\left\{(e^x, e^p, Q) \in (\mathbb{C}^*)^3 \colon \mathcal{A} \text{ has augmentation}\right\}.$$

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- The trefoil T:

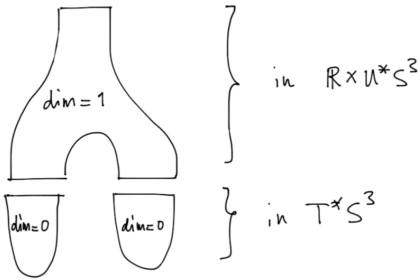
$$\begin{aligned} A_T(e^x, e^p, Q) = & (e^{4p} - e^{3p})e^{2x} \\ & + (e^{4p} - Qe^{3p} + 2Q^2 e^{2p} - 2Qe^{2p} - Q^2 e^p + Q^2)e^x \\ & + (-Q^3 e^p + Q^4). \end{aligned}$$

Augmentations and exact Lagrangian fillings

- Exact Lagrangian fillings L of Λ_K in T^*S^3 induces augmentations by

$$\epsilon(a) = \sum_{|a|=0} |\mathcal{M}_A(a)| A.$$

The map on coefficients are just the induced map on homology.



Augmentations and exact Lagrangian fillings

- There are two natural exact fillings of Λ_K: L_K and $M_K \approx S^3 - K$. Thus, $e^p = 1$ and $e^x = 1$ belong to $V_K|_{Q=1}$ for any K.

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- For the unknot $A_U(e^x, e^p, Q = 1) = (1 - e^x)(1 - e^p)$.

- For the trefoil $A_T(e^x, e^p, Q = 1) = (1 - e^x)(1 - e^p)(e^{3p} - 1)$.

Augmentations and exact Lagrangian fillings

- Looking at the exact Lagrangian fillings L_K and M_K more closely one shows that for $Q = 1$, the degree 0 knot contact homology of K can be described in terms of the knot group $\pi = \pi_1(S^3 - K)$ as an algebra over $\mathbb{C}[e^{\pm x}, e^{\pm p}]$ freely generated by the elements of π subject to the following relations:
  - $[e] = 1 - e^p$
  - If l is the longitude loop and m the meridian loop then $[l\gamma] = [\gamma l] = e^x[\gamma]$ and $[m\gamma] = [\gamma m] = e^p[\gamma]$.
  - $[\gamma_1\gamma_2] - [\gamma_1 m\gamma_2] - [\gamma_1][\gamma_2] = 0$.

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- Using this one checks that if $\rho \colon \pi \to GL(n)$ is a representation with

$$\rho(m) = \mathrm{diag}(e^{p_0}, 1, \dots, 1), \quad \rho(l) = \mathrm{diag}(e^{x_0}, *, \dots, *),$$

then (e^{x_0}, e^{p_0}) lies in the augmentation variety.

- For $SU(2)$-representations, $\rho'(m) = \text{diag}(e^p, e^{-p})$, $\rho'(l) = \text{diag}(e^x, e^{-x})$, $\rho(\gamma) = e^{\text{lk}(\gamma,K)}\rho'(\gamma)$ in $GL(2)$ satisfies the above with $e^{p_0} = e^{2p}$ and $e^{x_0} = e^x$ thus

$$(e^{2p} - 1)\, A\text{-polynomial}(K)(e^p, e^x) \mid A_K(e^{2p}, e^x)$$

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$$(e^{2p} - 1)\, A\text{-polynomial}(K)(e^p, e^x) \mid A_K(e^{2p}, e^x)$$

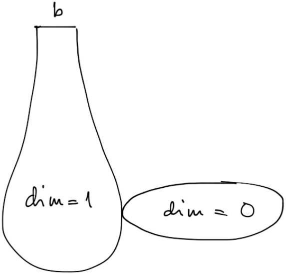
- In fact all augmentations for $Q = 1$ arises from such flat $GL(n)$ connections by a result of Cornwell.

Augmentations and non-exact Lagrangian fillings

- The Lagrangian conormal can be shifted off of the 0-section. It becomes non-exact but a similar version of compactness still holds: closed disks lie in a compact, and disks at infinity are as for $\Lambda_K \times \mathbb{R}$.

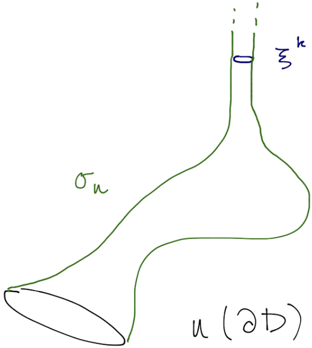
Augmentations and non-exact Lagrangian fillings

- The definition of a chain map in the exact case does not work because of new boundary phenomena.



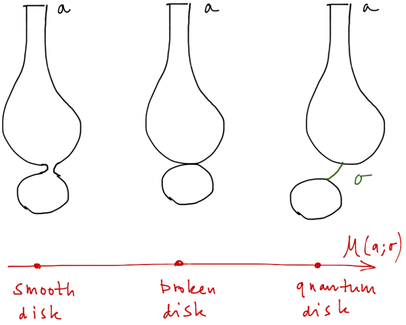
Compare the family of real curves in $\mathbb{C}^2$, $xy = \epsilon$, $\epsilon \to 0$.

- We resolve this problem by using bounding chains: fix a chain σ_D for each rigid disk D that connects its boundary in L_K to a multiple of a standard homology generator at infinity.



Augmentations and non-exact Lagrangian fillings

- We introduce quantum corrected holomorphic disks with punctures: these are ordinary holomorphic disks with all possible insertions of σ along the boundary. In the moduli space $\mathcal{M}_A(a;\sigma)$ of quantum corrected disks, boundary bubbling become interior points.

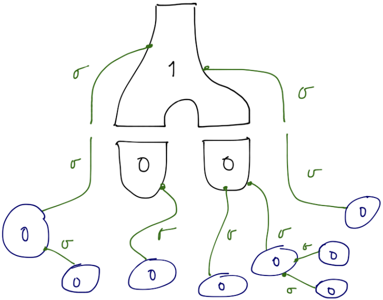


Augmentations and non-exact Lagrangian fillings

- Analyzing the boundary then shows that

$$\epsilon(a) = \sum_{|a|=1} \mathcal{M}_A(a;\sigma)e^A$$

is a chain map provided $p = \frac{\partial W_K}{\partial x}$.



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- Here W_K is the Gromov-Witten disk potential which is a sum over weighted trees, where there is a holomorphic disk at each vertex of the tree and where each edge is weighted by the linking number of the boundaries of the disks at the vertices of its endpoints, as measured by the bounding chains.

Many component links

- For an m-component link, the GW-disk potential of the conormal filling sees only the individual knot components. We thus need other Lagrangian fillings of the conormal tori. Such fillings indeed exist in special cases. In the general case there are candidates. The GW-disk potential W_K of such a filling gives a Lagrangian (locally $x_j = \frac{\partial W}{\partial p_j}$) augmentation variety V_K in $(\mathbb{C}^*)^m \times (\mathbb{C}^*)^m$.

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- Even for knots the augmentation variety is generally not irreducible and one would like to have weakly unobstructed Lagrangian fillings in Y for every component.